Modeling And Pricing Event Risk
Part I - Annualized Cash Flow

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In this white paper we will build a model to calculate the expected value of annualized cash flow at some future time t given the probability that a one-time event that materially reduces annualized cash flow may occur in the future. To assist us in this endeavor we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with calculating the expected value of go-forward annualized cash flow for a company whose major customer currently represents 60% of revenue. Our client estimates that the average duration of a customer relationship is 8 years.

Table 1: Model Assumptions

Symbol	Description	Value
C_0	Annualized cash flow at time zero (in dollars)	1,000,000
L	Average duration of customer relationships (in years)	8.00
g	Year over year expected cash flow growth rate (%)	5.00
ω	Reduction in cash flow if major customer leaves (%)	60.00

We will use our model to answer the following questions:

Question 1: What is expected annualized cash flow at the end of year five?
Question 2: What is the mean and variance of annualized cash flow log return at the end of year five? (1)
Question 3: What is the probability that annualized cash flow at the end of year five is less than $500,000?

(1) Log return is comprised of the random cash flow growth rate minus the probability-weighted jump (i.e. major customer leaves). Because annualized cash flow is log-normally distributed an approximation for annualized cash flow at time t would be...

$$\text{Approximation: } C_t \approx C_0 \,\text{Exp}\left\{\text{mean} + \frac{1}{2}\,\text{variance}\right\} \tag{1}$$

Jump Arrival Time

We will define the variable N_t to be a jump counter that counts the number of jumps that are realized over the finite time interval $[0, t]$. We will assume that once we realize the first jump at time t then there will not be another jump over the time interval $[t, \infty]$. In other words we will cap the number of jumps over the infinite time interval $[0, \infty]$ at one such that...

$$N_t = \text{Number jumps realized over time interval } [0, t] \text{ ...where... } N_t \in \left\{0, 1\right\} \tag{2}$$

We will define the variable ω to be jump size and the variable J to be the log of one minus jump size. For our purposes jump size represents the percent of annualized cash flow that is permanently lost when the jump event occurs. Using Equation (2) above we will make the following definitions...

$$\omega = \text{Jump size ...where... } 0 \leq w \leq 1 \text{ ...and... } J = \ln\left(1 - \omega\right) \text{ ...such that... Exp}\left\{J N_t\right\} = (1 - \omega)^{N_t} \tag{3}$$

We will define the variable λ to be the average number of jumps that we expect to realize over any given time interval $[t, t+1]$. Given that we defined the variable L in Table 1 above as the average duration of a customer relationship, the equation for λ is...

$$\lambda = \mathbb{E}\Bigg[\text{number of annual jumps}\Bigg] = \frac{1}{L} \tag{4}$$

It is important to note that λ has no memory in that the fact that a jump did not occur over the time interval $[0, t]$ does not influence the expected future arrival time of the jump from the perspective of time t. In other words, the actual duration of the past relationship with the customer has no influence on the expected duration of the future relationship.

We will define the variable τ to be the arrival time of the jump in years. If we assume that the arrival time of the jump is exponentially-distributed then using Equation (4) above the mean and variance of the distribution of jump arrival times are... [1]

$$\text{Arrival time mean} = \mathbb{E}\Bigg[\tau\Bigg] = \frac{1}{\lambda} \text{ ...and... } \text{Arrival time variance} = \mathbb{E}\Bigg[\tau^2\Bigg] - \Bigg(\mathbb{E}\Bigg[\tau\Bigg]\Bigg)^2 = \frac{1}{\lambda^2} \tag{5}$$

We will view time as a continuum that runs from negative infinity to positive infinity. We need a reference point so we will call the current point in time [time zero] and assume that the jump event did not occur prior to time zero. We want to determine the probability of a jump occurring or not occurring over the time interval $[0, t]$. Using Equations (2), (4) and (5) above the equation for the probability that the jump event will not occur over the time interval $[0, t]$ from the perspective of time zero is... [1]

$$\text{Probability of no jump} = \text{Prob}\Bigg[\tau > t\Bigg] = \text{Prob}\Bigg[N_t = 0\Bigg] = \text{Exp}\Bigg\{-\lambda\, t\Bigg\} \tag{6}$$

The probability that the jump event will occur is equal to one minus the probability that the jump event will not occur. Using Equation (6) above the equation for the probability that the jump event will occur over the time interval $[0, t]$ from the perspective of time zero is... [1]

$$\text{Probability of a jump} = \text{Prob}\Bigg[\tau \le t\Bigg] = \text{Prob}\Bigg[N_t = 1 \,\Bigg|\, N_0 = 0\Bigg] = 1 - \text{Exp}\Bigg\{-\lambda\, t\Bigg\} \tag{7}$$

Annualized Cash Flow

Using the model parameters in Table 1 above we will make the following definitions...

$$\mu = \ln\Bigg(1 + g\Bigg) - \frac{1}{2}\sigma^2 \text{ ...where... } \sigma = v \tag{8}$$

We will define the variable C_t to be annualized cash flow at time t, the variable δW_t to be the change in a geometric Brownian motion, and the variable δN_t to be the change in the jump count variable. Using Equations (2), (3) and (8) above, the stochastic differential equation that defines the change in annualized cash flow at time t from the perspective of time zero is...

$$\delta C_t = \mu\, C_t\, \delta t + \sigma\, C_t\, \delta W_t - \omega\, C_t\, \delta N_t \text{ ...where... } \delta W_t \sim N\Bigg[0, \delta t\Bigg] \text{ ...and... } \delta N_t \in \Bigg\{0, 1\Bigg\} \tag{9}$$

The solution to Equation (9) above is the equation for random annualized cash flow at time t from the perspective of time zero, which is...

$$C_t = C_0\, \text{Exp}\Bigg\{\mu\, t + \sigma\sqrt{t}\, Z_t + J N_t\Bigg\} \text{ ...where... } Z_t \sim N\Bigg[0, 1\Bigg] \text{ ...and... } N_t \in \Bigg\{0, 1\Bigg\} \tag{10}$$

Note that using Equation (3) above we can rewrite Equation (10) above as...

$$C_t = C_0\, \text{Exp}\Bigg\{\mu\, t + \sigma\sqrt{t}\, Z_t\Bigg\}\, \text{Exp}\Bigg\{J N_t\Bigg\} = (1 - \omega)^{N_t}\, C_0\, \text{Exp}\Bigg\{\mu\, t + \sigma\sqrt{t}\, Z_t\Bigg\} \tag{11}$$

Using Equation (11) above the equation for the expected value of annualized cash flow at time t from the perspective of time zero is...

$$\mathbb{E}\bigg[C_t\bigg] = \mathbb{E}\bigg[(1-\omega)^{N_t}\, C_0 \,\text{Exp}\bigg\{\mu\, t + \sigma\sqrt{t}\, Z_t\bigg\}\bigg] \tag{12}$$

Since the random variables N_t and Z_t are independent (by definition) we can rewrite Equation (12) above as...

$$\mathbb{E}\bigg[C_t\bigg] = C_0\, \mathbb{E}\bigg[(1-\omega)^{N_t}\bigg]\, \mathbb{E}\bigg[\text{Exp}\bigg\{\mu\, t + \sigma\sqrt{t}\, Z_t\bigg\}\bigg] \tag{13}$$

Using the jump probabilities in Equations (6) and (7) above, and given that the random variable N_t can be zero or one, the expected value of the first half of Equation (13) above is...

$$\begin{aligned}
\mathbb{E}\bigg[(1-\omega)^{N_t}\bigg] &= \bigg[(1-\omega)^0 \times \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg] + \bigg[(1-\omega)^1 \times \bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg)\bigg] \\
&= \text{Exp}\bigg\{-\lambda\, t\bigg\} + (1-\omega) \times \bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg) \\
&= \text{Exp}\bigg\{-\lambda\, t\bigg\} + 1 - \text{Exp}\bigg\{-\lambda\, t\bigg\} - \omega\bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg) \\
&= 1 - \omega\bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg)
\end{aligned} \tag{14}$$

Note that the second half of Equation (13) above is the expected value of a lognormally-distributed random variable. Using the moment generating function of the lognormal distribution the expected value of the second half of Equation (13) above is... [2]

$$\mathbb{E}\bigg[\text{Exp}\bigg\{\mu\, t + \sigma\sqrt{t}\, Z_t\bigg\}\bigg] = \text{Exp}\bigg\{\mu\, t + \frac{1}{2}\,\sigma^2 t\bigg\} \tag{15}$$

If we combine Equations (14) and (15) above the solution to Equation (12) above is...

$$\mathbb{E}\bigg[C_t\bigg] = C_0\,\bigg[1 - \omega\bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg)\bigg]\,\text{Exp}\bigg\{\mu\, t + \frac{1}{2}\,\sigma^2 t\bigg\} \tag{16}$$

Change in the Log of Annualized Cash Flow

We will define the variable R_t to be the change in the log of annalized cash flow over the time interval $[0, t]$. Using Equation (10) above the equation for R_t is...

$$R_t = \ln\bigg(C_t\bigg) - \ln\bigg(C_0\bigg) \tag{17}$$

We will define the variable F_t to be the log of annualized cash flow at time t. Using Equation (10) above the equation for F_t is...

$$\begin{aligned}
F_t &= \ln\bigg(C_t\bigg) \\
&= \ln\bigg(C_0\bigg) + \mu\, t + \sigma\sqrt{t}\, Z_t + J N_t \\
&= F_0 + \mu\, t + \sigma\sqrt{t}\, Z_t + J N_t
\end{aligned} \tag{18}$$

Using Equation (18) above we can rewrite Equation (17) above as...

$$R_t = F_t - F_0 = \mu\, t + \sigma\sqrt{t}\, Z_t + J N_t \tag{19}$$

Using Appendix Equations (37) and (39) below the equation for the first moment of the distribution of R_t is...

$$\begin{aligned}
\mathbb{E}\bigg[R_t\bigg] &= \mathbb{E}\bigg[\mu\, t + \sigma\sqrt{t}\, Z_t + J N_t\bigg] \\
&= \mu\, t + \sigma\sqrt{t}\,\mathbb{E}\bigg[Z_t\bigg] + J\,\mathbb{E}\bigg[N_t\bigg] \\
&= \mu\, t + J\bigg(1 - \text{Exp}\bigg\{-\lambda\, t\bigg\}\bigg)
\end{aligned} \tag{20}$$

Using Appendix Equations (38), (40) and (41) below the equation for the second moment of the distribution of R_t is...

$$\begin{aligned}
\mathbb{E}\bigg[R_t^2\bigg] &= \mathbb{E}\bigg[\bigg(\mu\,t+\sigma\sqrt{t}\,Z_t+J N_t\bigg)^2\bigg]\\
&= \mathbb{E}\bigg[\mu^2 t^2+\sigma^2 t\,Z_t^2+J^2 N_t^2+2\,\mu\,t\,\sigma\sqrt{t}\,Z_t+2\,\mu\,t\,J N_t+2\,\sigma\sqrt{t}\,Z_t\,J N_t\bigg]\\
&= \mu^2 t^2+\sigma^2 t\,\mathbb{E}\bigg[Z_t^2\bigg]+J^2\,\mathbb{E}\bigg[N_t^2\bigg]+2\,\mu\,t\,\sigma\sqrt{t}\,\mathbb{E}\bigg[Z_t\bigg]+2\,\mu\,t\,J\,\mathbb{E}\bigg[N_t\bigg]+2\,\sigma\sqrt{t}\,J\,\mathbb{E}\bigg[Z_t N_t\bigg]\\
&= \mu^2 t^2+\sigma^2 t+J^2\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)+2\,\mu\,t J\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)
\end{aligned}\tag{21}$$

The mean of the distribution of R_t is equal to the first moment of the distribution of R_t. Using Equation (20) above the equation for the mean of the distribution of R_t is...

$$\text{Mean of } R_t=\mathbb{E}\bigg[R_t\bigg]=\mu\,t+J\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\tag{22}$$

The variance of the distribution of R_t is equal to the second moment of the distribution of R_t minus the square of the first moment of the distribution or R_t. Using Equations (20) and (21) above the equation for the variance of the distribution or R_t is...

$$\begin{aligned}
\text{Variance of } R_t &= \mathbb{E}\bigg[R_t^2\bigg]-\bigg[\mathbb{E}\bigg[R_t\bigg]\bigg]^2\\
&= \bigg[\mu^2 t^2+\sigma^2 t+J^2\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)+2\,\mu\,t J\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)\bigg]\\
&\quad-\bigg[\mu^2 t^2+2\,\mu\,t J\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)+J^2\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)^2\bigg]\\
&= \sigma^2 t+J^2\bigg[\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)-\bigg(1-\text{Exp}\bigg\{-\lambda\,t\bigg\}\bigg)^2\bigg]\\
&= \sigma^2 t+J^2\bigg[1-\text{Exp}\bigg\{-\lambda\,t\bigg\}-1+2\,\text{Exp}\bigg\{-\lambda\,t\bigg\}-\text{Exp}\bigg\{-2\,\lambda\,t\bigg\}\bigg]\\
&= \sigma^2 t+J^2\bigg[\text{Exp}\bigg\{-\lambda\,t\bigg\}-\text{Exp}\bigg\{-2\,\lambda\,t\bigg\}\bigg]
\end{aligned}\tag{23}$$

Threshold Probabilities

We want to determine the probability that random annualized cash flow at time t is greater than or less than some given threshold value. We will define the variable X_t to be the threshold value at time t. Our first task is to determine the value of the normally-distributed random variable Z_t in Equation (10) above such that $C_t = X_t$. Using Equation (10) above the equation for Z_t is...

$$X_t=C_0\,\text{Exp}\bigg\{\mu\,t+\sigma\sqrt{t}\,Z_t+J N_t\bigg\}\ \text{...when...}\ Z_t=\bigg[\ln\bigg(\frac{X_t}{C_0}\bigg)-\mu\,t-J N_t\bigg]\bigg/\sigma\sqrt{t}\tag{24}$$

Note that in Equation (24) above the random variable N_t in the right side of that equation can be either one or zero given that there was or was not a jump over the time interval $[0,t]$. We want to break Z_t in Equation (24) above into two parts. We will define the variable Z_t^Y to be the value of the random variable Z_t given that there was a jump and Z_t^N to be the value of the random variable Z_t in Equation (24) given that there was not a jump. The two equations are...

$$Z_t^Y=\bigg[\ln\bigg(\frac{X_t}{C_0}\bigg)-\mu\,t-J\bigg]\bigg/\sigma\sqrt{t}\ \text{...and...}\ Z_t^N=\bigg[\ln\bigg(\frac{X_t}{C_0}\bigg)-\mu\,t\bigg]\bigg/\sigma\sqrt{t}\tag{25}$$

Using Equation (25) above, and noting that $N[z]$ is the cumulative normal distribution function for a standardized normally-distributed random variable, the equation for the probability that random annualized cash flow at time t

is less a given threshold value is...

$$\text{Prob}\Big[C_t < X_t\Big] = \text{Prob}\Big[N_t = 1\Big] \times N\Big[Z_t^Y\Big] + \text{Prob}\Big[N_t = 0\Big] \times N\Big[Z_t^N\Big] \tag{26}$$

Using Equation (26) above the equation for the probability that random annualized cash flow at time t is greater than a given threshold value is...

$$\text{Prob}\Big[C_t > X_t\Big] = 1 - \text{Prob}\Big[C_t < X_t\Big] \tag{27}$$

The Answers To Our Hypothetical Problem

Using Equation (4) above and the model parameters from Table 1 above the value of λ is...

$$\lambda = \frac{1}{8} = 0.1250 \tag{28}$$

Using Equation (8) above and the model parameters from Table 1 above the values of μ and σ are...

$$\mu = \ln\Big(1 + 0.05\Big) - \frac{1}{2} \times 0.25^2 = 0.0175 \;\; \text{...where...} \;\; \sigma = 0.25 \tag{29}$$

Using the problem definition above the value of ω is...

$$\omega = \text{Jump size} = 0.60 \;\; \text{...where...} \;\; J = \ln(1 - 0.60) = -0.9163 \tag{30}$$

Question 1: What is expected annualized cash flow at the end of year five?

Using Equation (16) and the values of the model parameters above expected annualized cash flow at the end of year five is...

$$\mathbb{E}\Big[C_5\Big] = 1{,}000{,}000 \times \Bigg[1 - 0.60\Bigg(1 - \text{Exp}\Bigg\{-0.1250 \times 5\Bigg\}\Bigg)\Bigg] \text{Exp}\Bigg\{0.0175 \times 5 + \frac{1}{2} \times 0.25^2 \times 5\Bigg\} = 920{,}400 \tag{31}$$

Question 2: What is the mean and variance of annualized cash flow log return at the end of year five?

Using Equations (22) above the mean of log return at the end of year five is...

$$\text{mean} = 0.0175 \times 5 - 0.9163\Bigg(1 - \text{Exp}\Bigg\{-0.1250 \times 5\Bigg\}\Bigg) = -0.3381 \tag{32}$$

Using Equations (23) above the variance of log return at the end of year five is...

$$\text{variance} = 0.25^2 \times 5 + (-0.9163)^2\Bigg[\text{Exp}\Bigg\{-0.1250 \times 5\Bigg\} - \text{Exp}\Bigg\{-2 \times 0.1250 \times 5\Bigg\}\Bigg] = 0.5214 \tag{33}$$

Question 3: What is the probability that annualized cash flow at the end of year five is less than \$500,000?

Using Equation (25) above the values of the random variables Z_t given a jump and no jump are...

$$Z_5^Y = \Bigg[\ln\Bigg(\frac{500{,}000}{1{,}000{,}000}\Bigg) - 0.0175 \times 5 + 0.9163\Bigg] \Bigg/ \Bigg(0.25 \times \sqrt{5}\Bigg) = 0.2427 \tag{34}$$

$$Z_5^N = \Bigg[\ln\Bigg(\frac{500{,}000}{1{,}000{,}000}\Bigg) - 0.0175 \times 5\Bigg] \Bigg/ \Bigg(0.25 \times \sqrt{5}\Bigg) = -1.396 \tag{35}$$

Using Equations (6), (7) and (26) above the answer to the question is...

$$\text{Prob}\Big[C_5 < 500{,}000\Big] = 0.4647 \times 0.5959 + 0.5353 \times 0.08129 = 0.3203 \tag{36}$$

References

[1] Gary Schurman, *Modeling Exponential Arrival Times,* September, 2015.

[2] Gary Schurman, *The Lognormal Distribution,* June, 2015.

Appendix

A. The equation for the expected value of the standardized, normally-distributed random variable Z_t in log return Equation (20) above is...

$$\mathbb{E}\left[Z_t\right] = 0 \text{ by definition} \tag{37}$$

B. The equation for the expected value of the square of the standardized, normally-distributed random variable Z_t in log return Equation (20) above is...

$$\mathbb{E}\left[Z_t^2\right] = 1 \text{ by definition} \tag{38}$$

C. The equation for the expected value of the exponentially-distributed random variable N_t in log return Equation (20) above is...

$$\mathbb{E}\left[N_t\right] = \left[0 \times \mathrm{Exp}\left\{-\lambda\,t\right\}\right] + \left[1 \times \left(1 - \mathrm{Exp}\left\{-\lambda\,t\right\}\right)\right] = 1 - \mathrm{Exp}\left\{-\lambda\,t\right\} \tag{39}$$

D. The equation for the expected value of the square of the exponentially-distributed random variable N_t in log return Equation (20) above is...

$$\mathbb{E}\left[N_t^2\right] = \left[0^2 \times \mathrm{Exp}\left\{-\lambda\,t\right\}\right] + \left[1^2 \times \left(1 - \mathrm{Exp}\left\{-\lambda\,t\right\}\right)\right] = 1 - \mathrm{Exp}\left\{-\lambda\,t\right\} \tag{40}$$

E. The equation for the expected value of the product of the standardized, normally-distributed random variable Z_t and the exponentially-distributed random variable N_t in log return Equation (21) above is...

$$\mathbb{E}\left[Z_t\,N_t\right] = 0 \text{ because the random variables are independent} \tag{41}$$

F. The solution to the following integral given that $x < 0$ is...

$$\begin{aligned}\int\limits_0^\infty \mathrm{Exp}\left\{x\,t\right\}\delta t &= \frac{1}{x}\,\mathrm{Exp}\left\{x\,t\right\}\Big[_0^\infty \\ &= \frac{1}{x}\left(\mathrm{Exp}\left\{x \times \infty\right\} - \mathrm{Exp}\left\{x \times 0\right\}\right) \\ &= -\frac{1}{x}\end{aligned} \tag{42}$$

G. Using Appendix Equation (42) above the solution to the following integral given that $x < 0$ is...

$$\begin{aligned}\int\limits_0^a \mathrm{Exp}\left\{x\,t\right\}\delta t + b\int\limits_a^\infty \mathrm{Exp}\left\{x\,t\right\}\delta t &= \frac{1}{x}\,\mathrm{Exp}\left\{x\,t\right\}\Big[_0^a + b\,\frac{1}{x}\,\mathrm{Exp}\left\{x\,t\right\}\Big[_a^\infty \\ &= \frac{1}{x}\left(\mathrm{Exp}\left\{x \times a\right\} - \mathrm{Exp}\left\{x \times 0\right\}\right) + b\,\frac{1}{x}\left(\mathrm{Exp}\left\{x \times \infty\right\} - \mathrm{Exp}\left\{x \times a\right\}\right) \\ &= \frac{1}{x}\left(\mathrm{Exp}\left\{x\,a\right\} - 1\right) + b\,\frac{1}{x}\left(0 - \mathrm{Exp}\left\{x\,a\right\}\right) \\ &= \frac{1}{x}\left(\mathrm{Exp}\left\{x\,a\right\} - 1 - b\,\mathrm{Exp}\left\{x\,a\right\}\right) \\ &= \frac{1}{x}\left((1-b)\,\mathrm{Exp}\left\{x\,a\right\} - 1\right)\end{aligned} \tag{43}$$